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XII. *A Conjecture concerning the Method by which Cardan's Rules for resolving the Cubic Equation $x^3 + qx = r$ in all cases (or in all magnitudes of the known quantities q and r) and the Cubic Equation $x^3 - qx = r$ in the first Case of it (or when r is greater than $\frac{29\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) were probably discovered by Scipio Ferreus, of Bononia, or whoever else was the first Inventor of them. By Francis Maferes, Esq. F. R. S. Curfitor Baron of the Exchequer.*

Read January 27, 1780.

A R T I C L E I.

THERE is nothing more amusing, or more grateful to an inquisitive mind, in the study of the sciences of Geometry and Algebra (for if we banish from it the ridiculous mysteries arising from the supposition of *negative* quantities, or quantities *less than nothing*, the latter may deserve the name of a *science* as well as the former) than to contemplate the methods by which the several ingenious and surprizing truths that are delivered in the
books

books that treat of them were first discovered. This we are sometimes enabled to do by the authors themselves to whom we are indebted for these discoveries, who have candidly informed their readers of the several steps, and sometimes of the accidents, by which they have been led to them: but it also often happens, that the authors of these discoveries have neglected to give their readers this satisfaction, and have contented themselves with either barely delivering the propositions they have found out, without any demonstrations, or with giving formal and positive demonstrations of them, which command indeed the assent of the understanding to their truth, but afford no clue whereby to discover the train of reasoning by which they were first found out; and consequently contribute but little to enable the reader to make similar discoveries himself on the like subjects. This seems to be the case with those ingenious rules for the resolution of certain cubic equations, which are usually known by the name of CARDAN's rules. We are told to make certain substitutions of some quantities for others in these equations $x^3 + qx = r$ and $x^3 - qx = r$ (which are the objects of those rules) and certain suppositions concerning the quantities so substituted; by doing which we find, that those equations will be transformed into other equations which will involve the sixth power of the unknown quantity

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tity contained in them, but which (though of double the dimensions of the original equations $x^3+qx=r$ and $x^3-qx=r$, from which they were derived) will be more easy to resolve than those equations, because they will contain only the sixth power and the cube of the unknown quantity which is their root, and consequently will be of the same form as quadratic equations; so that by resolving them as quadratic equations we may obtain the value of the cube of the unknown quantity which is their root, and afterwards, by extracting the cube-root of the said value, we may obtain the value of the said root, or unknown quantity, itself; and then at last, by the relation of this last root to x , or the root of the original equation, (which relation is derived from the suppositions that have been made in the course of the preceding transformations) we may determine the value of x . And, if we please to examine the several steps of this process with sufficient attention, we may perceive, as we go along, that all these substitutions are legitimate and practicable, or are founded upon possible suppositions; though I cannot but observe, that the writers on algebra, for the most part, have not been so kind as to shew us that they are so. But still the question recurs, “ How came SCIPIO “ FERREUS, of Bononia (who, as CARDAN tells us, was “ the first inventor of these rules) or the other person, “ whoever

“ whoever he was, that invented them, to think of making these lucky substitutions which thus transform the original cubic equations into equations of the sixth power which contain only the sixth and third powers of the unknown quantities which are their roots, and consequently are of the form of quadratic equations?” To answer this question as well as I can *by conjecture* (for I know of no historical account of this matter in any book of algebra) and in a manner that appears to me to be *probable*, is the design of the following pages.

2. The most probable conjecture concerning the invention of these rules, called CARDAN's rules, by SCIPIO FERREUS, of Bononia, or whoever else was the inventor of them, seems to be this: that the said inventor tried a great variety of methods of reducing the three cubic equations of the third class, to wit, $x^3 + qx = r$ and $x^3 - qx = r$, and $qx - x^3 = r$ (to some one of which all other cubic equations may, by proper substitutions, be reduced) to a lower degree, or to a more simple form, by substituting various quantities in the stead of x , in hopes that some of the terms arising by such substitutions might be equal to others of them, and, having contrary signs prefixed to them, might destroy them, and thereby render the new equation more simple and manageable than the old one. And, amongst other trials, it seems natural to

imagine, that he would substitute the sum or difference of two other quantities instead of x , as being the most simple and obvious substitutions that could be made. And by making these substitutions, the above mentioned rules would of course come to be discovered, as well as the aforesaid limitation of them in the resolution of the equation $x^3 - qx = r$, which restrains the rule to those cases only in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and their utter inutility in all the cases of the equation $qx - x^3 = r$. This will appear by examining each of these equations separately in the following manner.

Of the equation $x^3 + qx = r$.

ART. 3. In the equation $x^3 + qx = r$ the investigator of these rules would naturally be inclined to substitute the difference of two quantities (which we will here call y and z , and of which we will suppose y to be the greater) instead of x , rather than their sum, or would suppose x to be equal to $y - z$, rather than to $y + z$; because, if he was to suppose x to be equal to the sum of the two quantities y and z , and was to substitute that sum, or the binomial quantity $y + z$, instead of x in the equation $x^3 + qx = r$, it is evident, that (as the signs of x^3 and qx

are, both of them, affirmative) the terms of the new equation, arising from such substitution, would all of them be likewise affirmative; and consequently none of them, though they should happen to be exactly equal to each other, could exterminate each other, and thereby render the new equation more simple than the old one, which was the only view with which the substitution would have been made. He would, therefore, suppose x to be equal to $y-z$; and by substituting this quantity instead of x in the original equation $x^3+qx=r$, he would transform that equation into the following one, to wit,

$$y^3-3yyz+3yzx-x^3+qy-qx=r,$$

$$\text{or } y^3-3yz \times \overline{y-z} - x^3 + q \times \overline{y-z} = r.$$

Now in this equation it is evident, that the terms $3yz \times \overline{y-z}$ and $q \times \overline{y-z}$ have contrary signs; and therefore, if their co-efficients $3yz$ and q can be supposed to be equal to each other, those terms will mutually destroy each other, and the equation will be reduced to the following short one, $y^3-x^3=r$. And if in this equation we substitute, instead of x , its value $\frac{q}{3y}$, derived from the same supposition of the equality of q and $3yz$, the equation will be $y^3-\frac{q^3}{27y^3}=r$; and, by multiplying both sides by y^3 , it will be $y^6-\frac{q^3}{27}=ry^3$; which equation, though it rises to the sixth power of the unknown quantity y , is

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evidently of the form of a quadratic equation, and may therefore be resolved, so far as to find the value of the cube of y , in the same manner as a quadratic equation; after which it will be possible to find the value of y itself by the mere extraction of the cube root; and then at last, from the relation of y to x (derived from the foregoing suppositions that $y - x$ was equal to x , and that $3yx$ was equal to q , and consequently x equal to $\frac{q}{3y}$) we shall be able to determine the value of x .

Art. 4. It would therefore remain for the investigator of this method to inquire, whether or no the supposition, "that $3yx$ was equal to q ," was a possible supposition; that is, whether it was possible (whatever might be the magnitudes of q and r) for two quantities, y and x , to exist, whose nature should be such that their difference $y - x$ should be equal to the unknown quantity x in the equation $x^3 + qx = r$, and that three times their product should at the same time be equal to q , or their simple product to the third part of q . And this supposition he would soon find to be always possible, whatever may be the magnitudes of q and r ; because, if the lesser quantity x is supposed to increase from *o ad infinitum*, and the greater quantity y is likewise supposed to increase with equal swiftness, or to receive equal increments in the same

times, and thereby to preserve their difference $y - x$ always of the same magnitude, or equal to x , it is evident that the product or rectangle yx will increase continually at the same time from *o ad infinitum*, and consequently will pass successively through all degrees of magnitude, and therefore must at one point of time during its increase become equal to $\frac{q}{3}$.

And having thus found this supposition of the equality of yx and $\frac{q}{3}$, or of $3yx$ and q , to be always possible, whatever might be the magnitudes of q and r , our investigator would justly consider his solution of the equation $x^3 + qx = r$ (which was founded on that supposition) as legitimate and complete. And thus we see in what manner it seems probable, that CARDAN's rule for resolving the cubic equation $x^3 + qx = r$ may have been discovered.

Of the equation $x^3 - qx = r$.

Art. 5. In this second equation $x^3 - qx = r$, in which the second term qx is subtracted from the first, or marked with the sign $-$, it seems to have been natural for the person who invented these rules to substitute the *sum* as well as the *difference* of two other quantities, y and x , instead

stead of x , in the terms x^3 and qx , in hopes of such an extermination of equal terms, and consequential reduction of the equation to one of a simpler and more manageable form, as was found to be so useful in the case of the former equation $x^3+qx=r$. We will therefore try both these substitutions; and, as that of the difference $y-z$ has in the former case proved so successful, we will begin by that.

Art. 6. Now, by substituting the difference $y-z$ instead of x in the equation $x^3-qx=r$, we shall transform it into the following equation, to wit, $y^3-3yyz+3yz^2-z^3-q \times \sqrt{y-z}=r$, or $y^3-3yz \times \sqrt{y-z}-z^3-q \times \sqrt{y-z}=r$; in which the terms $3yz \times \sqrt{y-z}$ and $q \times \sqrt{y-z}$ have both of them the same sign - prefixed to them, and consequently can never exterminate each other, whether $3yz$ be equal or unequal to q . This substitution therefore is in this case of no use.

Art. 7. We will now therefore try the substitution of the *sum* of y and z , instead of their difference, in the equation $x^3-qx=r$.

Now, if x be supposed to be equal to $y+z$, and $y+z$ be substituted instead of it in the equation $x^3-qx=r$, that equation will be thereby transformed into the following one, to wit,

$$y^3 +$$

$$y^3 + 3yyz + \overline{3yzx + z^3 - q \times y + z} = r,$$

$$\text{or } y^3 + 3yz \times y + z + z^3 - q \times y + z = r.$$

Now in this equation, the terms $3yz \times \sqrt{y+z}$ and $q \times \sqrt{y+z}$ have contrary signs. Consequently, if they can be supposed to be equal to each other, they will destroy each other, and the equation will be thereby reduced to the following short one, $y^3 + z^3 = r$; that is, if $3yz$ and q can be supposed to be equal to each other, or if yz can be supposed to be equal to $\frac{q}{3}$, the equation will be reduced to the short equation $y^3 + z^3 = r$. And, if in this short equation we substitute, instead of z , its value $\frac{q}{3y}$ (derived from the same supposition of the equality of $3yz$ and q) the equation thence resulting will be $y^3 + \frac{q^3}{27y^3} = r$; and by multiplying both sides by y^3 , it will be $y^6 + \frac{q^3}{27} = ry^3$; and, by subtracting y^6 from both sides, it will be $ry^3 - y^6 = \frac{q^3}{27}$; which, though it rises to the sixth power of y , is evidently of the form of a quadratic equation, and consequently may be resolved in the same manner as a quadratic equation, so far as to find the value of y^3 , or the cube of the root y ; after which it will be possible to find the value of y itself by the mere extraction of the cube root; and, lastly, from the relation of y to z (contained in the two suppositions, that $y+z$ is equal to x , and that

$3yz$ is equal to q , and consequently that x is equal to $\frac{q}{3y}$) we may determine the value of x .

Art. 8. The only thing, therefore, that would remain for the investigator of these rules to do, in order to know whether the foregoing method of resolving the equation $x^3+qx=r$ was practicable or not, would be to inquire, whether it was possible in all cases, that is, in all magnitudes of the known quantities q and r , for $3yz$ to be equal to q , or for yz (or the product or rectangle of the two quantities y and z , whose sum is equal to x) to be equal to $\frac{q}{3}$, and, if it was not possible in all cases, but only in some, to determine in what cases it *was* possible, or what must be the relation between q and r to make it possible.

Art. 9. Now, in order to determine this question, it would be proper and natural to observe, that the quantity yz , or the product of the two quantities y and z , whose sum is supposed to be equal to x , can never be greater than the square of half that sum, that is, than the square of $\frac{x}{2}$, or than $\frac{xx}{4}$, by El. 2, 5, but may be of any magnitude that does not exceed that square. Therefore, if $\frac{q}{3}$ is greater than $\frac{xx}{4}$, it will be impossible for yz to be equal to it; but, if $\frac{q}{3}$ is either equal to, or less than,

$$\frac{xx}{4},$$

$\frac{xx}{4}$, it will be possible for yx to be equal to it, and if $\frac{q}{3}$ is exactly equal to $\frac{xx}{4}$, x will be exactly equal to y , and each of them equal to one half of x . We must, therefore, inquire what is the magnitude of x when $\frac{q}{3}$ is equal to $\frac{xx}{4}$. Now, when $\frac{xx}{4} = \frac{q}{3}$, xx will be $= \frac{4q}{3}$, and $x = \frac{2\sqrt{q}}{\sqrt{3}}$: therefore, when x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be impossible for yx to be equal to $\frac{q}{3}$; but when x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be possible for yx to be equal to $\frac{q}{3}$.

But when x is $= \frac{2\sqrt{q}}{\sqrt{3}}$, x^3 will be $= \frac{8q\sqrt{q}}{3\sqrt{3}}$, and qx will be $= \frac{2q\sqrt{q}}{\sqrt{3}}$ or $\frac{6q\sqrt{q}}{3\sqrt{3}}$, and consequently $x^3 - qx$ will be $= \frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{6q\sqrt{q}}{3\sqrt{3}} = \frac{2q\sqrt{q}}{3\sqrt{3}}$.

Therefore, if it be true (as we shall presently see that it is) that while x increases from being equal to \sqrt{q} (which is evidently its least possible magnitude) to any other magnitude, the compound quantity $x^3 - qx$, or the excess of x^3 above qx , will also continually increase from 0 (to which it is equal when x is $= \sqrt{q}$, or xx is $= q$) to some correspondent magnitude without ever decreasing; it will follow that, when x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $x^3 - qx$ will be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and when

x is

x is greater than $\frac{2\sqrt{q}}{\sqrt{q}}$, the compound quantity $x^3 - qx$ will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and, *à converſo*, if the compound quantity $x^3 - qx$ is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be leſs than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if the compound quantity $x^3 - qx$ is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$. Conſequently, if the compound quantity $x^3 - qx$, or, its equal, the abſolute term r in the equation $x^3 - qx = r$, is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, it will be impoſſible for yz to be equal to $\frac{q}{3}$; but, if $x^3 - qx$, or r , is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be poſſible for yz to be equal to $\frac{q}{3}$. Therefore, if $x^3 - qx$, or r , is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, the foregoing method of reſolving the cubic equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx = r$, or r , is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be practicable.

Art. 10. It now only remains to be proved, that while x increaſes, from being equal to \sqrt{q} , *ad infinitum*, the compound quantity $x^3 - qx$ will likewiſe increaſe from 0 *ad infinitum*, without ever decreaſing. Now this may be demonſtrated as follows.

Art. 11. It is evident, that while x increafes from being equal to \sqrt{q} *ad infinitum*, both the quantities x^3 and qx will increafe *ad infinitum* likewise. But it does not therefore follow, that the excefs of x^3 above qx will continually increafe at the fame time. This will depend upon the relation of the contemporary increments of x^3 and qx ; if the increment of x^3 in any given time is equal to the contemporary increment of qx , the compound quantity $x^3 - qx$ will neither increafe nor decreafe, but continue always of the fame magnitude during the faid time, notwithstanding the increafe of x ; if the former increment is lefs than the latter, the faid compound quantity will decreafe; and, if it is greater, it will increafe. We muft therefore inquire, whether the increment of x^3 in any given time is greater or lefs than the contemporary increment of qx .

Art. 12. Now, if \dot{x} be put for the increment which x receives in any given time, the increment of x^3 in the fame time will be the excefs of $\overline{x+\dot{x}}^3$ above x^3 , that is, the excefs of $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ above x^3 ; and the increment of qx in the fame time will be the excefs of $q \times \overline{x+\dot{x}}$, or $qx + q\dot{x}$, above qx ; that is, the increment of x^3 will be $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, and that of qx will be $q\dot{x}$. Now in the equation $x^3 - qx = r$ it is evident, that xx muft be greater than q ; for otherwife x^3 would not be greater than qx ,

as it is supposed to be. Consequently, $x \times x$ must be greater than qx ; and, *à fortiori*, $3x^2x + 3xx^2 + x^3$ (which is more than triple of x^2x) must be greater than qx ; that is, the increment of x^3 will be greater than the contemporary increment of qx . Therefore, the excess of x^3 above qx , or the compound quantity $x^3 - qx$, will increase continually, without decreasing, while x increases from $\sqrt[3]{q}$ *ad infinitum*. Q. E. D.

Art. 13. It follows, therefore, upon the whole of these inquiries, that if the compound quantity $x^3 - qx$, or, its equal, the absolute term r , is less than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, it will be impossible for yz to be equal to $\frac{q}{3}$, and consequently the foregoing method of resolving the equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx$ or r is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be possible for yz to be equal to $\frac{q}{3}$, and consequently, the foregoing method of resolving the equation $x^3 - qx = r$ will be practicable. And thus we see in what manner it is probable that CARDAN'S rule for resolving the cubic equation $x^3 - qx = r$ in the first case of it, or when r is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, together with the restriction of it to that first case, may have been discovered.

Of the Equation $qx - x^3 = r$.

Art. 14. In the third equation $qx - x^3 = r$ the terms x^3 and qx have different signs, as well as in the second equation $x^3 - qx = r$; and therefore it seems to have been natural for the inventor of CARDAN's rules to try both the substitutions of $y - z$ and $y + z$ instead of x in this equation, as well as in that second equation, in hopes of an extermination of equal terms that are marked with contrary signs, and a consequent reduction of the equation to another which, though of double the dimensions of the equation $qx - x^3 = r$, should have been of a simpler form and more easy to be resolved. But it will be found upon trial, that neither of these substitutions will answer the end proposed.

Art. 15. For, in the first place, let us suppose x to be $= y - z$. Then we shall have $x^3 = y^3 - 3yyz + 3yyz - z^3 = y^3 - 3yz \times \overline{y - z} - z^3$, and $qx = q \times \overline{y - z}$, and consequently $qx - x^3 = q \times \overline{y - z} - y^3 + 3yz \times \overline{y - z} + z^3$. Therefore, $q \times \overline{y - z} - y^3 + 3yz \times \overline{y - z} + z^3$ will be $= r$. Now in this equation it is evident, the terms $q \times \overline{y - z}$ and $3yz \times \overline{y - z}$ have the same signs, and therefore can never destroy each other. Therefore, no such method of resolving this equation

$qx - x^3 = r$ as was found above for the two former equations $x^3 + qx = r$ and $x^3 - qx = r$, can be obtained by substituting the difference $y - z$ in it instead of x .

Art. 16. We will now try the substitution of $y + z$ instead of x in the terms of this equation.

Now, if x be supposed to be $= y + z$, we shall have $x^3 = y^3 + 3yyz + 3yzx + z^3 = y^3 + 3yz \times \overline{y + z} + z^3$, and $qx = q \times \overline{y + z}$, and consequently, $qx - x^3 = q \times \overline{y + z} - y^3 - 3yz \times \overline{y + z} - z^3$. Therefore, $q \times \overline{y + z} - y^3 - 3yz \times \overline{y + z} - z^3$ will be $= r$.

In this equation it is true indeed that the terms $q \times \overline{y + z}$ and $3yz \times \overline{y + z}$ have different signs. But they cannot be equal to each other: for, since the three terms y^3 and $3yz \times \overline{y + z}$ and z^3 are all marked with the sign $-$, or are to be subtracted from the first term $q \times \overline{y + z}$, and the remainder is $= r$, it is evident, that $q \times \overline{y + z}$ must be greater than the sum of all the three terms y^3 , $3yz \times \overline{y + z}$, and z^3 , taken together, and therefore, *à fortiori*, greater than $3yz \times \overline{y + z}$ alone. Therefore, no such extermination of equal terms marked with contrary signs as took place in the transformed equations derived from the two former equations $x^3 + qx = r$ and $x^3 - qx = r$, can take place in this transformed equation derived from the equation $qx - x^3 = r$ by substituting $y + z$ in its terms instead of x ; and consequently no such method of resolving the equation

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tion $qx - x^3 = r$ as has been found for the resolution of the equations $x^3 + qx = r$ and $x^3 - qx = r$, can be obtained by means of that substitution.

Art. 17. These are the methods of investigation by which I conceive it to be probable, that CARDAN's rules for the resolution of the cubic equations $x^3 + qx = r$ and $x^3 - qx = r$, together with the limitation of the rule relating to the latter of those equations, and their inapplicability to the third equation $qx - x^3 = r$, may have been discovered by the first inventor of them.

